Solution 11

Supplementary Problems

1. Let S be the triangle with vertices at $(1,0,0),(0,2,0),(0,0,7)$ with normal pointing upward. Verify Stokes' theorem for the vector field $\mathbf{F} = x\mathbf{i} + 3z\mathbf{j}$.

   Solution. The equation for the triangle is given by $ax+by+cz=d$. To determine the coefficients, we note $(0,2,0)-(1,0,0)=(-1,2,0)$ and $((0,0,7)-(1,0,0)=(-1,0,7)$ and the vector $(-1,2,0)\times(-1,0,7)=(14,7,2)$ is in the normal direction of the surface. So the equation is given by $14x+7y+2z=d$. Since $(1,0,0)$ is on the surface, $14\times 0+0+0=d$, $d=14$. The equation of the plane containing the triangle is $14x+7y+2z=14$. The normal pointing upward, that is, the z-component is positive, is $(14,7,2)$.

   We have $\nabla\mathbf{F}=-3\mathbf{i}$. The surface is the graph of $z=(14-14x-7y)/2$ over the triangle T with vertices at $(0,0),(1,0),(0,2)$. Then $\mathbf{r}_x=(1,0,-7)$ and $\mathbf{r}_y=(0,1,-7/2)$ so $\mathbf{r}_x\times\mathbf{r}_y = 7\mathbf{i}+7/2\mathbf{j}+\mathbf{k}$. We have

$$\begin{aligned}\iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma &= \iint_T (-3\mathbf{i})\cdot(7\mathbf{i}+7/2\mathbf{j}+\mathbf{k})\,dA \\ &= -21\iint_T dA \\ &= -21.\end{aligned}$$

   We conclude

$$\iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = -21. \tag{1}$$

   On the other hand, the boundary of S consists of three line segments C_1,C_2,C_3. First, C_1 from $(1,0,0)$ to $(0,2,0)$ which is parametrized by $(x(t),y(t),z(t))=(1,0,0)+t(-1,2,0)=(1-t,2t,0), t\in[0,1]$. We have

$$\int_{C_1}\mathbf{F}\cdot\, d\mathbf{r} = \int_0^1 (x(t)\mathbf{i}+3z(t)\mathbf{j})\cdot(-\mathbf{i}+2\mathbf{j})\,dt = -\int_0^1(1-t)\,dt = -1/2\ .$$

   Next, C_2 from $(0,2,0)$ to $(0,0,7)$ is parametrized by $(0,2-27,7t), t\in[0,1]$. As before we get

$$\int_{C_2}\mathbf{F}\cdot\, d\mathbf{r} = -21\ .$$

   Finally, C_3 from $(0,0,7)$ to $(1,0,0)$ is parametriized by $(t,0,7-7t), t\in[0,1]$. We have

$$\int_{C_3}\mathbf{F}\cdot\, d\mathbf{r} = 1/2\ .$$

   It follows that

$$\int_C \mathbf{F}\cdot\, d\mathbf{r} = \left(\int_{C_1}+\int_{C_2}+\int_{C_3}\right)\mathbf{F}\cdot\, d\mathbf{r} = -1/2-21+1/2=-21\ ,$$

   which is equal to (1).

2. Show that for a closed oriented surface S, that is, a surface without boundary,

$$\iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = 0\ .$$

Hint: See how to apply Stokes' theorem.

Solution. Choose a point on S and draw a sphere with center at this point. When the radius of the sphere is sufficiently small, the intersection of the sphere with S is a simple closed curve C. Call the part of S outside C S_1 and the inside S_2. Then S is the union of S_1 and S_2 with common boundary C. Denote the oriented boundary with the correct orientation with S_1 by C and the oriented boundary with the correct direction with S_2 be C'. C and C' are the same curve but with opposite orientation. We have

$$\begin{aligned}\iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma &= \left(\iint_{S_1}+\iint_{S_2}\right)\nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma \\ &= \int_C \mathbf{F}\cdot d\mathbf{r}+\int_{C'}\mathbf{F}\cdot d\mathbf{r} \\ &= \int_C \mathbf{F}\cdot d\mathbf{r}-\int_C \mathbf{F}\cdot d\mathbf{r} \\ &= 0\ .\end{aligned}$$

3. (Optional) Let S be the surface given by $(x,y)\mapsto(x,y,f(x,y)),(x,y)\in D$. That is, it is the graph of f over the region D. Show that in this case Stokes' theorem

$$\iint_S \nabla\times\mathbf{F}\,d\sigma=\oint_C \mathbf{F}\cdot d\mathbf{r}\ ,$$

($\mathbf{F}$ is a smooth vector field on S) can be deduced from Green's theorem for some vector field on D. Hint: Let the boundary of D be $\mathbf{r}(t)=(x(t),y(t))$. Then the boundary of S is $\mathbf{c}(t)=(x(t),y(t),f(x(t),y(t)))$. Convert the integration in S and C to the integration on D and the boundary of D respectively.

Solution. Use $(x,y)\mapsto(x,y,f(x,y))$ to parametrize S (what else?). The upward normal is given by

$$\mathbf{n}=\frac{(-f_x,-f_y,1)}{\sqrt{1+f_x^2+f_y^2}}\ ,$$

so

$$\iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma=\iint_D[(P_y-N_z)(-f_x)-(P_x-M_z)(-f_y)+(N_x-M_y)]\,dA(x,y)\ , \quad (2)$$

where the curl of $\mathbf{F}$ is evaluated at $(x,y,f(x,y))$. On the other hand, let γ be the boundary of D parametrized by $(x(t),y(t)),t\in[a,b]$. Then C is parametrized by $(x(t),y(t),f(x(t),y(t)))$, so its velocity is $(x'(t),y'(t),f_xx'(t)+f_yy'(t))$. We have

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \int_a^b[Mx'(t)+Ny'(t)+P(f_xx'(t)+f_yy'(t))]\,dt \\ &= \int_a^b[(M+Pf_x)x'(t)+(N+Pf_y)y'(t)]\,dt\ ,\end{aligned}$$

where $\mathbf{F}$ is evaluated at $(x(t),y(t),f(x(t),y(t)))$. Letting $R=M+Pf_x$ and $Q=N+Pf_y$ (evaluating at $(x,y,f(x,y))$ and applying Green's theorem to the vector field $R\mathbf{i}+Q\mathbf{j}$ on

D, we have

$$
\begin{aligned}
& \int_a^b [(M + Pf_x)x'(t) + (N + Pf_y)y'(t)]\, dt \\
= & \oint_\gamma (Rx'(t) + Qy'(t))\, dt \\
= & \oint_\gamma (R\mathbf{i} + Q\mathbf{j}) \cdot \, d\mathbf{r} \\
= & \iint_D (\frac{\partial Q}{\partial x} - \frac{\partial R}{\partial y})\, dA \\
= & \iint_D [(N_x + N_z f_x + (P_x + P_z f_x) f_y + P f_{yx}) - (M_y + M_z f_y + (P_y + P_z f_y) f_x + P f_{xy})]\, dA \\
= & \iint_D (N_x + N_z f_x + P_x f_y - M_y - M_z f_y - P_y f_x)\, dA \ ,
\end{aligned}
$$

which is equal to (2). Stokes' theorem holds in this case.